G. Leitmann*
U.S. Naval Ordnance Test Station

Optimum Thrust Programing for High-Altitude Rockets

A discussion of the problem of establishing the thrust-time relation which will achieve the optimum compromise between reduction in gravity and drag losses and thereby result in minimum fuel expenditure.

Introduction

One of the most important exterior ballistic problems associated with high-altitude rocketry is that of carrying a specified pay load to a desired height most economically—i.e., with minimum expenditure of fuel. The two main deterrents to achieving altitude are the forces of gravity and aerodynamic drag, both of which diminish with increasing altitude. Unfortunately, the requirements for reducing the deterrent effects of gravity and drag are antithetical. Gravity losses are proportional to flight time whereas drag losses are proportional to some power of the velocity. Thus, diminution of gravity losses requires a short flight time—i.e., a high velocity—whereas reduction of drag losses calls for low velocity. One of the problems which confronts the designer of a high-altitude rocket, then, is the establishment of an optimum thrust program—the thrust-time relation which will achieve the optimum compromise between reduction in gravity and drag losses and thereby result in minimum fuel expenditure.

The American rocket pioneer R. H. Goddard in his paper "A Method of Reaching Extreme Altitudes"[1] was first in calling attention to the problem of thrust programing. For purposes of mathematical analysis, Goddard considered an idealized rocket—namely, a right circular cone, pay load at the tip, and casing which drops away continuously (with zero velocity with respect to the remaining rocket) as the burning surface recedes (see Fig. 1). Goddard inferred the existence of a solution to his problem from the following argument: "If, at any intermediate altitude, the velocity of ascent be very great, the air resistance (depending on the square of the velocity) will also be very great. On the other hand, if the velocity of ascent be very small, force will be required to overcome gravity for a long period of time. In both cases the mass necessary to be expelled will be excessively large. Evidently, then, the velocity of ascent must have a special value at each point in space." Goddard went on to state that the determination of the necessary velocity-time function presents a new and unsolved problem in the calculus of variations. Consequently, he abandoned a rigorous approach and constructed an approximate, numerical solution. Almost a decade after the publication (1919) of Goddard's classical paper, Hamel,[2] in a very brief publication, objected to the lack of rigor in Goddard's analysis and pointed out the existence of a solution by means of the calculus

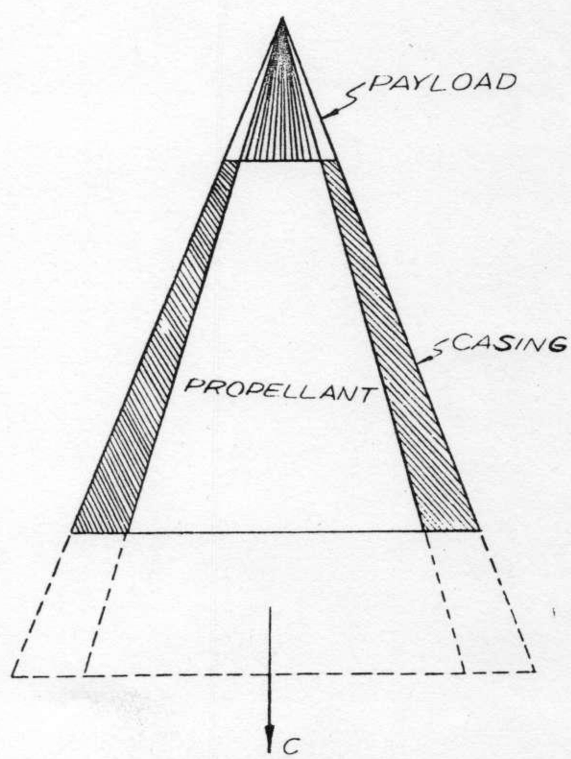


Fig. 1. Goddard's idealized rocket.

* Head, Aeroballistics Analysis Section, China Lake, Calif.

of variations. More recently, other workers—e.g., Tsien and Evans,[3] Lawden,[4] and Leitmann[5,6]—have derived necessary conditions for the existence of minimum initial mass of a rocket required to transport a given burnout mass to specified altitude.

Theory

Let m be the mass of the rocket and s the altitude at time t. A dot will denote differentiation with respect to time. It is assumed that the aerodynamic drag D is a function of s and $\dot{s}$ only. The effective exhaust velocity c and the acceleration of gravity g are taken as constant. (Taking account of the change of g with altitude modifies the form of the solution slightly.[6]) Conditions at the outset of powered flight, $t = 0$, will be denoted by subscript zero; conditions at burnout, $t = t_1$, by subscript one. Symbols + and − attached to the subscripts will denote conditions at the instant succeeding $t = 0$ and preceding $t = t_1$, respectively. Coordinate $s(t)$ is assumed continuous over the whole trajectory. Functions $\dot{s}(t)$ and $m(t)$ are taken continuous over the interval $0 < t \leq t_1$. The discontinuity of $\dot{s}$ and m at $t = 0$—i.e., impulsive boosting at launch—is permitted. (As a matter of fact, it will be shown that impulsive boosting is a requirement of the solution.) At the outset of powered flight, $t = 0$, $s = s_0 = 0$, $\dot{s} = \dot{s}_0 = 0$, and $m = m_0$. After initial impulsive boosting, $s = s_{0+} = s_0 = 0$ still, but $\dot{s} = \dot{s}_{0+}$ and $m = m_{0+}$. At burnout, $t = t_{1-}$, $s = s_{1-} = s_1$, $\dot{s} = \dot{s}_{1-} = \dot{s}_1$, and $m = m_{1-} = m_1$ (these functions being assumed continuous). However, the acceleration $\ddot{s}$ and the mass-flow rate $\dot{m}$ are not assumed continuous at burnout. Thus, at $t = t_{1-}$, $\ddot{s} = \ddot{s}_{1-}$ and $\dot{m} = \dot{m}_{1-}$. At the outset of coasting flight, $t = t_1$, $\ddot{s} = \ddot{s}_1$, and $\dot{m} = \dot{m}_1 = 0$.

The equation of rocket motion for vertical flight, neglecting earth's rotation and taking account of drag and gravity forces only, is

$$c\dot{m} + (\ddot{s} + g)m + D(s, \dot{s}) = 0 \qquad (1)$$

Integration of Eq. (1) over the initial boost period δ, where $\delta \to 0$, is

$$m_0 = m_{0+} \exp(\dot{s}_{0+}/c) \qquad (2)$$

The equation of coasting flight is merely Eq. (1) with $\dot{m} = 0$ (no thrust).

The problem is then: Given m_1, c, g, and the function $D(s, \dot{s})$, what is the function $s(t)$ in order that the rocket reach summit altitude S with minimum initial mass m_0? Integration of the equation of coasting flight between $s = s_1$ and $s = S$ ($\dot{s} = \dot{s}_1$ and $\dot{s} = 0$) yields a relation between s_1 and $\dot{s}_1$ symbolized by

$$\dot{s}_1 = \phi(s_1) \qquad (3)$$

By means of the calculus of variations it can be shown that the following relations must be satisfied in order that m_0 be a minimum. Throughout powered flight,

$$\epsilon \equiv \frac{\partial D}{\partial s} - \left(\ddot{s}\frac{\partial^2 D}{\partial \dot{s}^2} + \dot{s}\frac{\partial^2 D}{\partial s \partial \dot{s}} + \frac{\dot{s}}{c}\frac{\partial D}{\partial s} + \frac{2\ddot{s} + g}{c}\frac{\partial D}{\partial \dot{s}} + \frac{\ddot{s} + g}{c^2}D\right) = 0 \qquad (4)$$

At burnout, $t = t_{1-}$,

$$\dot{s}_1[(\partial D/\partial \dot{s}) + (D/c)]_{t=t_1} = m_1 g + D(s_1, \dot{s}_1) \qquad (5)$$

The problem of determining the optimum solution, the function $s(t)$ which corresponds to a stationary value of m_0, is then determinate. Eqs. (3) and (5) may be solved for s_1 and $\dot{s}_1$. Eq. (4) is a second-order differential equation in $s(t)$ for whose integration s_1 and $\dot{s}_1$ constitute initial values. Upon integration of Eq. (4) over $0 < t < t_1$, the values of t_1 and $\dot{s}_{0+}$ are obtained. Eq. (1) can then be integrated to give m_{0+}—i.e.,

$$m_{0+} = \exp\left(-\frac{\dot{s}_{0+}}{c}\right)\left[\int_{0+}^{t_{1-}} \frac{D}{c}\exp\left(\frac{\dot{s} + gt}{c}\right)dt + m_1 \exp\left(\frac{\dot{s}_1 + gt_1}{c}\right)\right] \qquad (6)$$

The value of initial rocket mass m_0 is now found from Eq. (2). The acceleration $\ddot{s}$ can be determined from Eq. (4), whereupon Eq. (1) yields the thrust

$$T = -c\dot{m} = m(\ddot{s} + g) + D \qquad (7)$$

It is easy to show[6] that Eq. (4) possesses a first integral

$$I \equiv \left[mg + D - \dot{s}\left(\frac{\partial D}{\partial \dot{s}} + \frac{D}{c}\right)\right]\exp\left(\frac{\dot{s} + gt}{c}\right) = 0 \qquad (8)$$

Eq. (8) leads to the immediate conclusion that $\dot{s}$ cannot vanish in the interval $0 < t < t_1$. Since $D = 0$ when $\dot{s} = 0$, but $mg \neq 0$, it follows that $\dot{s} \neq 0$. Hence $\dot{s}_{0+} \neq 0$, and impulsive boosting is always required.

Special Case

It is customary to express the aerodynamic drag force in the form

$$D = (1/2)\rho a C_D \dot{s}^2 \qquad (9)$$

where

ρ = air density
a = reference area
C_D = drag coefficient

The air density ρ is essentially a function of altitude s. The drag coefficient C_D is a function of Mach and Reynolds Numbers and is therefore a function of both altitude and velocity. A simple, idealized expression for drag results when C_D is taken as constant. A fair approximation to ρ is an exponential in s. With these assumptions

$$D = W \exp(-\alpha s)\dot{s}^2 \qquad (10)$$

$$W = (1/2)a\rho_0 C_D$$
$$\rho_0 = \text{sea-level air density}$$
$$\alpha = \text{constant}$$

If the expression for drag given by Eq. (1) is used, Eq. (4) becomes

$$\frac{\ddot{s}}{g} = \frac{V[V^2 + (1-\beta)V - 2\beta]}{\beta[V^2 + 4V + 2]} \qquad (11)$$

where $V = \dot{s}/c, \qquad \beta = g/\alpha c^2$

Eq. (11) may be integrated for $s(V)$ and $t(V)$—i.e.,

$$\alpha s = V - V_{0+} + \frac{\gamma}{2}\ln\frac{2V + (1-\beta) - \gamma}{2V + (1-\beta) + \gamma} \times \frac{2V_{0+} + (1-\beta) + \gamma}{2V_{0+} + (1-\beta) - \gamma} + \frac{3+\beta}{2}\ln\frac{V^2 + (1-\beta)V - 2\beta}{V_{0+}^2 + (1-\beta)V_{0+} - 2\beta} \qquad (12)$$

$$\frac{gt}{c} = \ln\frac{V_{0+}}{V} + \frac{\gamma}{2}\ln\frac{2V + (1-\beta) - \gamma}{2V + (1-\beta) + \gamma} \times \frac{2V_{0+} + (1-\beta) + \gamma}{2V_{0+} + (1-\beta) - \gamma} + \frac{1+\beta}{2}\ln\frac{V^2 + (1-\beta)V - 2\beta}{V_{0+}^2 + (1-\beta)V_{0+} - 2\beta} \qquad (13)$$

where $\gamma = \sqrt{(1-\beta)^2 + 8\beta}$

Substitution of Eqs. (12) and (13) in the integrated form of Eq. (1) results in

$$M = \frac{m}{m_1} = \exp\left(-V - \frac{gt}{c}\right)\left\{\frac{Wc^2}{m_1 g}\beta V_{0+} \times \exp(V_{0+})\,[V_{0+}^2 + (1-\beta)V_{0+} - 2\beta] \times \left[\frac{V+2}{V^2 + (1-\beta)V - 2\beta} - \frac{V_1 + 2}{V_1^2 + (1-\beta)V_1 - 2\beta}\right] + \exp\left(V_1 + \frac{gt_1}{c}\right)\right\} \qquad (14)$$

The thrust is then given by

$$\frac{T}{m_1 g} = \frac{Wc^2}{m_1 g}V^2\exp(-\alpha s) + \frac{m}{m_1}\left(1 + \frac{\ddot{s}}{g}\right) \qquad (15)$$

Eq. (5) takes the form

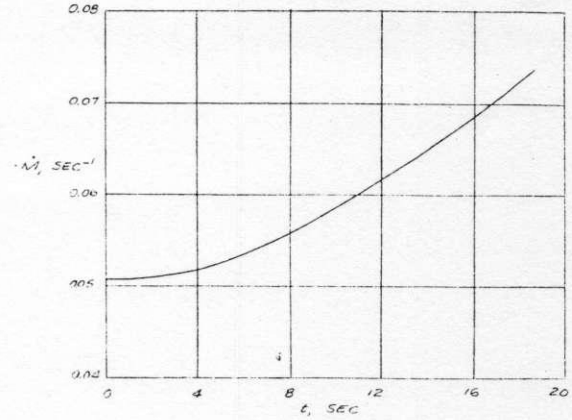
$$[illegible] = -2\beta\exp\left(2\beta\frac{Wc^2}{m_1 g}x_1\right) \times \int_{x_1}^{x_2}\exp\left(-2\beta\frac{Wc^2}{m_1 g}x\right)\frac{dx}{x} \qquad (16)$$

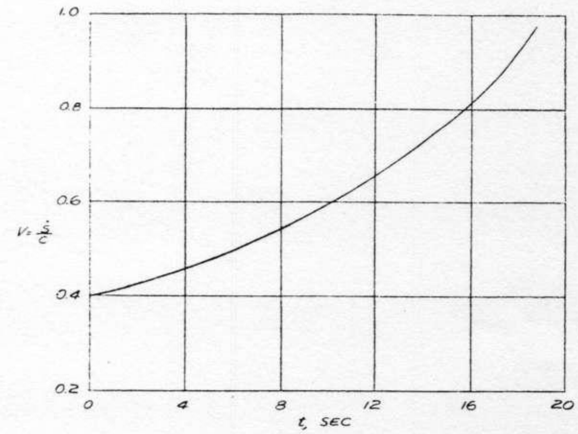
where

$$x_1 = \exp(-\alpha s_1), \qquad x_2 = \exp(-\alpha S)$$

The integral of Eq. (16) may be evaluated by means of the tabulated integral[7]

$$Ei(-y) = -\int_y^\infty \frac{\exp(-z)}{z}\,dz$$





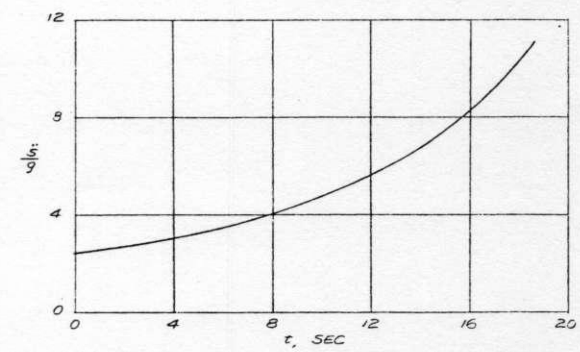


FIG. 2 (*Top*). Rate of mass flow as a function of time.
FIG. 3 (*Center*). Nondimensional velocity as a function of time.
FIG. 4 (*Bottom*). Acceleration as a function of time.

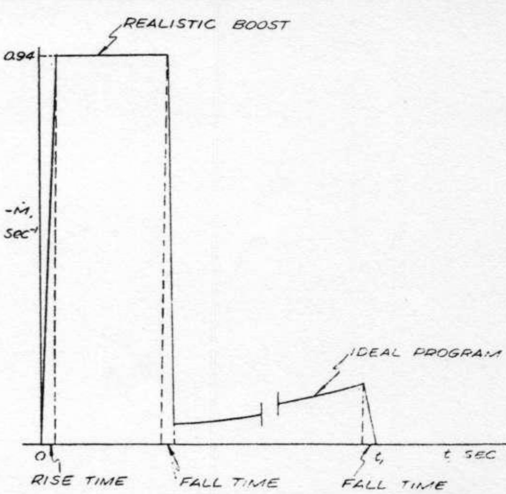


FIG. 5. Realistic thrust program.

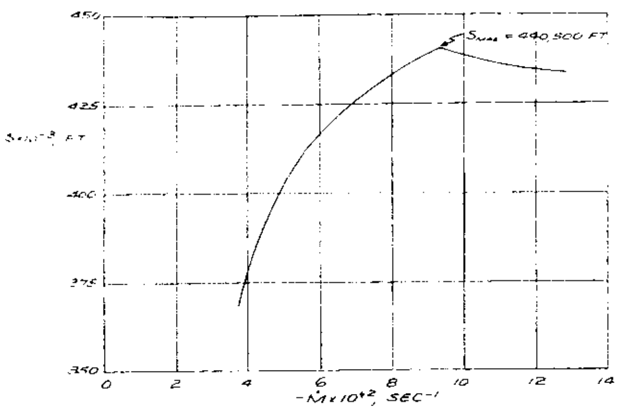


FIG. 6. Summit altitude as a function of mass-flow rate—constant thrust.

or from the series solution

$$\int \exp(-bx)\frac{dx}{x} = \ln|x| - \frac{bx}{1\cdot 1!} + \frac{(bx)^2}{2\cdot 2!} - \ldots$$

Finally, Eq. (5) becomes

$$(Wc^2/m_1g)\, V_1^2(1 + V_1) = \exp(\alpha s_1) \qquad (17)$$

The solution proceeds as follows: (a) Solve Eqs. (16) and (17) for s_1 and V_1; (b) substitute s_1 and V_1 for s and V, respectively, in Eq. (12) and solve for V_{0+}; (c) using this value of V_{0+} and setting $V = V_1$ in Eq. (13) yield t_1; (d) Eq. (14) then gives $m(V)$, and Eq. (15) with Eq. (11) is used to find $T(V)$. Since $t(V)$ is known, m and T can be expressed as functions of time t.

Numerical Example

Consider a rocket of the following characteristics:

$$W/m_1 = 10^{-5} \text{ ft.}, \qquad c = 5{,}500 \text{ ft./sec.}$$

required to reach a summit altitude

$$S = 92.6 \text{ miles} = 488{,}950 \text{ ft.}$$

with minimum fuel expenditure. Also

$$\alpha = 22{,}000 \text{ ft.}^{-1}, \qquad g = 32.2 \text{ ft./sec.}^2$$

Applying the equations of the preceding section results in

$$s_1 = 62{,}576 \text{ ft.}, \qquad \dot{s}_1 = 5{,}308 \text{ ft./sec.}$$
$$t_1 = 18.7 \text{ sec.}, \qquad \dot{s}_{0+} = 2{,}199 \text{ ft./sec.}$$
$$m_{0+}/m_1 = 2.10, \qquad m_0/m_1 = 3.14$$

The normalized mass-flow rate $\dot{M} = \dot{m}/m_1$, velocity $V = \dot{s}/c$, and acceleration $\ddot{s}/g$ are shown as functions of time in Figs. 2–4. The idealized thrust program requires the attainment of an initial velocity of 2,199 ft./sec. by means of an impulsive boost. In practice, the initial impulsive boost can be approximated by a short phase of high thrust. Thereafter, the ideal thrust program may be followed. To investigate the effect on summit altitude of replacing the ideal, impulsive boost by a realistic, high-thrust phase, consider the application of a constant, high thrust of duration 1.2 sec. during which the same amount of fuel, $m_0 - m_{0+}$, is consumed as in the ideal boost. Thereafter, burning proceeds according to program. Account is also taken of the finite time to build up and drop the thrust, with 0.1 sec. as the rise- and fall-times. The mass-flow rate for the high-thrust phase is quite attainable. For example, if the rocket under consideration has a burnout mass, m_1, of 10 slugs, the booster thrust is 51,700 lbs. Fig. 5 illustrates this realistic thrust program. For comparison, consider also the effect on summit altitude of keeping the thrust constant throughout burning (see Fig. 6). Integration of the equations of motion was carried out on an IBM 701 digital computer. The following summit altitudes resulted:

Thrust	Summit Altitude
Ideal program	488,950 ft.
Realistic program	471,680 ft.
Best constant thrust	440,500 ft.

Thus there is a loss of about 3.5 per cent in summit altitude in going from ideal to realistic thrust program. A further decrease of about 6.6 per cent in summit altitude is experienced when thrust is held constant at its best value.

Conclusion

It seems reasonable to conclude that the ideal thrust program can be approximated by a realistic one—one with a high-thrust phase in place of impulsive boost—without greatly affecting summit altitude. Whether or not the added complexity of controlling the thrust is worth the gain over best constant thrust is a problem which can only be answered in particular cases.

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