R-222

AN ANALYSIS OF THE MOTION OF LIQUIDS IN A PARTIALLY-FILLED ENCLOSURE

by

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ABSTRACT

In this note, the motion of the fluid in a partially filled enclosure is analyzed subject to the restrictions that viscous effects are negligible, that the total mass of fluid present and the local fluid density are both constants, and that the motion of the enclosure consists of small perturbations from a predetermined mean motion. The analysis is carried out using a frame fixed to the enclosure, thus making it relatively easy to treat cases involving complicated tank geometries. As an example, the behavior of fluid in a rigid, circularly cylindrical tank is investigated. The results are expressed as transfer functions applicable to missile dynamics.

LIST OF SYMBOLS

$\overline{A}$	Unperturbed acceleration of the origin of the body-fixed frame with respect to an inertially-fixed frame
a	Radius of a cylindrical tank
$\overline{a}_t$	Thrust acceleration = $\dfrac{\text{Thrust} - \text{Drag}}{\text{missile mass}}$
$C_1 \ldots C_{10}$	Constants of integration
$\overline{F}$	Force
f	Equation of the tank surface
$\overline{g}$	Gravity vector
h	Distance from the origin of the body-fixed axis to the unperturbed position of the free surface
J_n	Bessel function of the first kind of order n
k, n	Separation parameters
$\overline{M}$	Moment
m	Total mass of fluid in the tank
p	Ambient pressure in the fluid
$\overline{q}$	Apparent velocity of a fluid particle as seen in the body-fixed frame
q	Magnitude of $\overline{q}$
$\overline{R}$	Position vector in the body-fixed frame

s	Laplace transform variable
t	Time
$\overline{U}$	Velocity vector of a fluid particle as seen in the inertially-fixed frame
$\overline{1}_x, \overline{1}_y, \overline{1}_z$	Unit vectors defining the body-fixed frame
$\overline{V}$	Unperturbed velocity of origin of body-fixed frame with inertially-fixed frame
$\overline{\nu}$	Perturbation in $\overline{V}$
dVol	Differential element of volume
$x, y, z; r, \theta, x$	Coordinates of body-fixed frame
$\overline{\alpha}$	Perturbation in $\overline{A}$
ϕ	Velocity potential
ρ	Fluid mass density
ξ	Perturbation in the location of the free surface
$\overline{\omega}$	Angular velocity of body-fixed frame with respect to inertially-fixed frame

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1. Introduction and Assumptions

A. Introduction

The dynamics of liquid fuel missiles are affected significantly by the motion of the fuel in the tanks. This motion has been called "fuel sloshing". The effects have been studied by various scientists, and the major works in the field are included in the references at the end of this report. The fuel sloshing problem is a complex one and will continue to be an area of study for some time. This report is an additional effort at describing the forces and moments acting on a missile due to fuel sloshing.

The general problem is solved for arbitrary tank geometry subject to the assumptions dealt with below. Kinematical relationships between velocities in the fixed and moving frames are displayed. From Newton's second law and the assumption of zero vorticity, a pressure relationship (3-29) is obtained. The boundary conditions at the tank wall and at the free surface yield two relations, (3-28) and (3-33) which, with Laplace's equation, determine the flow pattern. The flow pattern and the pressure relation may then be integrated to find the forces and moments for a particular tank geometry and velocity potential.

In the example, boundary conditions for a rigid circular-cylindrical tank are incorporated into eq (3-28), (3-26), and (3-33). Laplace transforms of these equations are taken, and then a general solution is obtained in terms of Bessel functions and hyperbolic functions of eigenvalues. The constants are evaluated by means of the boundary conditions and orthogonality properties of eigenfunctions.

The Laplace transform (5-39) of the pressure relationship (3-29) is taken, and when the velocity potential, (5-38), is introduced, the expressions for forces and moments may be integrated. These results are written as four transfer functions:

$$\frac{\overline{F}_z}{\overline{\nu}_z}, \quad \frac{\overline{F}_z}{\overline{r}_y}, \quad \frac{\overline{M}_y}{\overline{\nu}_z}, \quad \text{and} \quad \frac{\overline{M}_y}{\overline{r}_y}$$

B. Assumptions

The assumptions made and their significance are listed below:

(1) The fluid is assumed to be incompressible. In most practical cases, the pressure gradient due to acceleration is not great enough to effect appreciable changes in the fluid density. This assumption removes the thermodynamic problems associated with the more general case, i.e., the energy equation (Bernoulli's equation) is found by integrating the momentum equation.

(2) The total mass of fluid in the tank is assumed to be constant. Except for local effects in the neighborhood of the "sink" or drain, the mass flow rates usually encountered in rocketry are small enough to preclude the appearance of serious errors arising from this source.

(3) The tank static pressure is great enough to prevent cavitation. Because this is a design criterion, this assumption seldom presents any difficulties.

(4) Surface tension forces are negligible. Here again, the errors introduced are local in scope. Often the ratio of inertia, or D'Alembert, forces to surface tension forces in the region of the free surface will be high, thus justifying this assumption.

(5) Viscous effects are neglected. When this assumption is warranted, it can be shown that an enormous mathematical simplification results, since the well developed tools of potential theory may be brought to bear on the problem. The necessary and sufficient condition for the

existence of a velocity potential is that the vorticity, i.e., the curl of the velocity, vanish everywhere in the fluid. In the case of an unbaffled tank oscillating with small amplitude, this condition is very nearly met because the vorticity contributions from skin friction on the tank walls and from splashing on the free surface are both small.

The purpose of baffles, it should be noted, is two-fold: first, some damping is added to the system; second, the natural frequencies of the various modes are altered. The damping effect arises when some of the kinetic energy of ordered motion is converted to kinetic energy of random eddying, whence it is dissipated by viscous effects. Changes in natural frequencies are due to altered tank geometry. In other words, a completely imperforate baffle would give rise to a totally different problem, which is, in many ways, easier to analyze. A good discussion of baffles may be found in reference (1).

(6) The tank motion is restricted to small disturbances from a zero lift trajectory. In addition to the reasons discussed above, this permits linearization of the boundary condition at the free surface. Also, variations in thrust acceleration are not permitted because the various natural frequencies are proportional to this quantity, and hence such a variation would give rise to non-linear effects. In the large majority of practical cases, the variation in thrust acceleration is small in the time interval of interest.

2. Kinematics

This report uses two frames, one fixed with respect to inertial space, the other fixed to the axis of symmetry of the tank. Although other schemes are feasible for uncomplicated tank geometries, this method is generally the simplest to apply in practice.

The kinematical relationships used are those derived in ref(2), or any similar text. They may be written as follows:

$$\overline{U} = \overline{V} + \overline{\nu} + \overline{q} + \overline{\omega} \times \overline{R}, \quad \text{and} \tag{2-1}$$

$$\frac{d\overline{U}}{dt} = \overline{A} + \overline{\alpha} + \frac{d\overline{q}}{dt} + \overline{\omega} \times \overline{q} + \frac{d}{dt}[\overline{\omega} \times \overline{R}]^{*} \tag{2-2}$$

Here the cross product is defined in the usual way. It should be remarked that in fluid mechanics the derivatives $\frac{d\overline{U}}{dt}$ and $\frac{d\overline{q}}{dt}$ are usually written as

$$\frac{d\overline{U}}{dt} = \frac{D\overline{U}}{Dt} = \frac{\partial\overline{U}}{\partial t} + [\overline{U} \bullet \nabla]\overline{U}, \quad \text{and} \tag{2-3}$$

$$\frac{d\overline{q}}{dt} = \frac{D\overline{q}}{Dt} = \frac{\partial\overline{q}}{\partial t} + [\overline{q} \bullet \nabla]\overline{q} \tag{2-4}$$

because the differentiation is performed while following a specific fluid particle. The expressions for such "substantial" derivatives may be found from the chain rule for partial derivatives.

As a part of the kinematical description of the problem an equation for the conservation of mass must be written. This equation, often called the "equation of continuity", is of the same form in both frames:

$$\nabla \bullet \overline{U} = 0, \quad \text{or} \tag{2-5}$$

$$\nabla \bullet \overline{q} = 0. \tag{2-6}$$

This result is valid only for an incompressible fluid, i.e.,

$$\rho = \text{constant}. \tag{2-7}$$

*The terms, $\overline{A} + \overline{\alpha}$, are the acceleration of the origin of the moving frame as measured in inertial space and then resolved into the appropriate components in the moving frame. These components would be $\frac{d\overline{V}}{dt} + \frac{d\overline{\nu}}{dt} + \overline{\omega} \times (\overline{V} + \overline{\nu})$ if all quantities were measured in the moving frame.

3. Dynamics

A. The Pressure Relationship

Newton's Second Law, when written for the body-fixed frame takes the form,

$$\bar{F} = (\rho dVol)\frac{d\bar{U}}{dt} = (\rho dVol)\bar{g} - \nabla p(dVol), \quad \text{or} \tag{3-1}$$

$$\frac{d\bar{U}}{dt} = \bar{g} - \frac{1}{\rho}\nabla p = \bar{A} + \bar{\alpha} + \frac{Dq}{Dt} + \bar{\omega}\times\bar{q} + \frac{d}{dt}[\bar{\omega}\times\bar{R}] \ . \tag{3-2}$$

Here only the principal diagonal terms in the stress tensor are used; the off-diagonal terms, which are due to viscosity, are important only in the boundary layer and such places. When these viscous terms are neglected, Kelvin's theorem states that the circulation around a closed contour consisting of a group of particles remains constant. If we presume zero initial circulation for all such paths, then it follows that the vorticity must always vanish,or

$$\nabla \times \bar{U} = 0 \ . \tag{3-3}$$

This in turn implies that $\bar{U}$-$\bar{V}$ is the gradient of a scalar potential, ϕ.

$$\bar{U} - \bar{V} = \bar{\nu} + \bar{q} + \bar{\omega}\times\bar{R} = \nabla\phi \ . \tag{3-4}$$

The author realizes that the above discussion is but a brief outline of some of the most important material in the field of hydrodynamics. For those interested in a more detailed discussion, the first five chapters of Lamb, ref(9), is considered to be one of the best dissertations on hydrodynamics available.

The next step is to manipulate equations (2-5), (2-6), (3-2), (3-3), and (3-4) in such a manner as to enable us to integrate equation (3-2) with respect to the three spatial coordinates.

From (3-3) and (2-1) it follows that

$$\nabla \times \overline{U} = \nabla \times \overline{V} + \nabla \times \overline{\nu} + \nabla \times \overline{q} + \nabla \times [\overline{\omega} \times \overline{R}] = 0 \ . \tag{3-5}$$

Noting that $\overline{V}$ and $\overline{\nu}$ are functions of time alone, $\nabla \times \overline{V}$ and $\nabla \times \overline{\nu}$ vanish, leaving,

$$\begin{aligned} \nabla \times \overline{q} &= -\nabla \times [\overline{\omega} \times \overline{R}] \qquad (3\text{-}6) \\ &= -[\nabla \bullet \overline{R}]\overline{\omega} + [\overline{\omega} \bullet \nabla]\overline{R} \\ &= -3\overline{\omega} + \overline{\omega} \ . \end{aligned}$$

$$\therefore \quad \nabla \times \overline{q} = -2\overline{\omega} \tag{3-7}$$

That is, the vorticity as seen in the body-fixed frame is such that it counterbalances the angular motion of the frame.

We next investigate the Coriolis acceleration term, $\overline{\omega} \times \overline{q}$.

$$2\overline{\omega} \times \overline{q} = -[\nabla \times \overline{q}] \times \overline{q} = \overline{q} \times [\nabla \times \overline{q}] . \tag{3-8}$$

Direct expansion of the x component (the other two are similar) shows that

$$2\overline{\omega} \times \overline{q} \Rightarrow \left\{ q_y \left[\frac{\partial q_y}{\partial x} - \frac{\partial q_x}{\partial y} \right] - q_z \left[\frac{\partial q_x}{\partial z} - \frac{\partial q_z}{\partial x} \right] \right\} \overline{1}_x . \tag{3-9}$$

Adding and subtracting $q_x \dfrac{\partial q_x}{\partial x}$ inside the bracket results in

$$2\overline{\omega} \times \overline{q} \Rightarrow \left\{ \left[q_x \frac{\partial q_x}{\partial x} + q_y \frac{\partial q_y}{\partial x} + q_z \frac{\partial q_z}{\partial x} \right] - \left[q_x \frac{\partial q_x}{\partial x} + q_y \frac{\partial q_x}{\partial y} + q_z \frac{\partial q_x}{\partial z} \right] \right\} \overline{1}_x , \tag{3-10}$$

or that

$$2\overline{\omega} \times \overline{q} = \nabla \left[\frac{1}{2} q^2 \right] - \left[\overline{q} \bullet \nabla \right] \overline{q} . \tag{3-11}$$

Noting that

$$\nabla \phi = \overline{\nu} + \overline{q} + \overline{\omega} \times \overline{R} , \tag{3-4}$$

equation (3-11) becomes

$$2\bar{\omega}\times\bar{q} = \nabla\left[\frac{1}{2}q^2\right] - [\bar{q}\bullet\nabla][\nabla\bar{\phi} - \bar{\nu} - \bar{\omega}\times\bar{R}] , \qquad \text{or} \tag{3-12}$$

$$2\bar{\omega}\times\bar{q} = \nabla\left[\frac{1}{2}q^2\right] - [\bar{q}\bullet\nabla][\nabla\phi - \bar{\omega}\times\bar{R}] .$$

Expansion of the term $[\bar{q}\bullet\nabla][\bar{\omega}\times\bar{R}]$ as before leads to

$$[\bar{q}\bullet\nabla][\bar{\omega}\times\bar{R}] \Rightarrow [q_x\frac{\partial}{\partial x} + q_y\frac{\partial}{\partial y} + q_z\frac{\partial}{\partial z}][\omega_y z - \omega_z y]\bar{1}_x \tag{3-13}$$

$$= [\omega_y q_z - \omega_z q_y]\bar{1}_x .$$

The expansion of the **y** and **z** components is similar.

$$\therefore [\bar{q}\bullet\nabla][\bar{\omega}\times\bar{R}] = \bar{\omega}\times\bar{q}. \tag{3-14}$$

The Coriolis acceleration now becomes

$$2\bar{\omega}\times\bar{q} = \nabla[\frac{1}{2}q^2] - [\bar{q}\bullet\nabla]\nabla\phi + [\bar{q}\bullet\nabla][\bar{\omega}\times\bar{R}] \tag{3-15}$$

$$= \nabla[\frac{1}{2}q^2] - [\bar{q}\bullet\nabla]\nabla\phi + \bar{\omega}\times\bar{q}, \quad \text{or}$$

$$\bar{\omega}\times\bar{q} = \nabla[\frac{1}{2}q^2] - [q\bullet\nabla]\nabla\phi . \tag{3-16}$$

Similar manipulations enable us to express the angular acceleration term, $\frac{d}{dt}[\bar{\omega}\times\bar{R}]$, as

$$\frac{d}{dt}[\bar{\omega}\times\bar{R}] = \frac{d}{dt}[\nabla\phi - \bar{\nu} - \bar{q}] , \quad \text{or} \tag{3-17}$$

$$\frac{d}{dt}[\bar{\omega}\times\bar{R}] = \frac{D}{Dt}[\nabla\phi - \bar{q}] + \bar{\omega}\times[\nabla\phi - \bar{q}] - \bar{\alpha} , \quad \text{or} \tag{3-18}$$

$$\frac{d}{dt}[\bar{\omega}\times\bar{R}] = \frac{\partial}{\partial t}\nabla\phi + [\bar{q}\bullet\nabla]\nabla\phi - \frac{D\bar{q}}{Dt} + \bar{\omega}\times[\nabla\phi - \bar{q}] - \bar{\alpha} . \tag{3 19}$$

We now substitute equations (3-16) and (3-19) into equation (3-9). It follows that

$$\bar{g} - \frac{1}{\rho} \quad \nabla p = \bar{A} + \bar{\alpha} + \frac{D\bar{q}}{Dt} + \bar{\omega} \times \bar{q} + \frac{\partial}{\partial t} \quad \nabla \phi + [\bar{q} \cdot \nabla] \nabla \phi \tag{3-20}$$

$$- \frac{D\bar{q}}{Dt} + \bar{\omega} \times [\nabla \phi - \bar{q}] - \bar{\alpha} \ , \quad \text{or}$$

$$- \frac{1}{\rho} \ \nabla p = \bar{A} - \bar{g} + \frac{\partial}{\partial t} \ \nabla \phi + [\bar{q} \bullet \nabla] \nabla \phi + \bar{\omega} \times \nabla \phi , \ \text{and} \tag{3-21}$$

$$- \frac{1}{\rho} \ \nabla p = \bar{A} - \bar{g} + \frac{\partial}{\partial t} \nabla \phi + \nabla \left[\frac{1}{2} q^2 \right] + \bar{\omega} \times [\nabla \phi - \bar{q}] . \tag{3-22}$$

Assuming interchangeability of the order of partial differentiation is valid, the momentum equation may be written as

$$- \frac{1}{\rho} \ \nabla p = \nabla \left[\frac{1}{2} q^2 \right] + \ \nabla \frac{\partial \phi}{\partial t} + \bar{A} - \bar{g} + \bar{\omega} \times [\bar{\nu} + \bar{\omega} \times \bar{R}] \ . \tag{3-23}$$

Since any body moving in a gravitational field has a term, $+\bar{g}$, contained in $\bar{A}$, it is apparent that the difference, $\bar{A} - \bar{g}$, is the total specific force, or in the case of a zero lift trajectory, the thrust acceleration, $\bar{a}_t$. With this result, equation (3-23) may be written in the form,

$$\nabla \left\{ \int \frac{dp}{\rho} + \frac{1}{2} q^2 + \frac{\partial \phi}{\partial t} + \bar{a}_t \bullet \bar{R} - \bar{\nu} \ \ (\bar{\omega} \times \bar{R}) - \frac{1}{2} \omega^2 R^2 + \frac{1}{2} (\bar{\omega} \bullet \bar{R})^2 \right\} = 0 \ . \tag{3-24}$$

After integration with respect to the three spatial coordinates, it is found that

$$p = - \rho \left\{ \frac{1}{2} q^2 + \frac{\partial \phi}{\partial t} + \bar{a}_t \bullet \bar{R} - \bar{\nu} \ \ (\bar{\omega} \times \bar{R}) - \frac{1}{2} \omega^2 R^2 + \frac{1}{2} (\bar{\omega} \bullet \bar{R})^2 + C(t) \right\} , \tag{3-25}$$

where C(t) is the function of integration. Since C(t) may be incorporated into ϕ without altering the flow pattern or the forces and moments exerted by the fluid on the tank walls, it is hereafter omitted.

B. The Flow Pattern

The solution for the flow pattern is found by solving Laplace's equation subject to certain boundary conditions. We obtain Laplace's equation by substituting equation (3-4) into equation (2-5). That is,

$$\nabla^2\phi = 0. \tag{3-26}$$

The boundary condition at the tank wall is that the fluid velocity normal to the tank wall is zero. $f(x,y,z,t) = 0$ is the equation of the tank wall, this condition may be written as

$$\bar{q} \cdot \nabla f = -\frac{\partial f}{\partial t} \quad , \qquad \text{or}^* \tag{3-27}$$

$$(\nabla\phi - \bar{\nu} - \bar{\omega} \times \bar{R}) \cdot \nabla f = -\frac{\partial f}{\partial t} \quad , \qquad \text{or}$$

$$\nabla\phi \cdot \nabla f = \bar{\omega} \times \bar{R} \cdot \nabla f + \bar{\nu} \cdot \nabla f - \frac{\partial f}{\partial t} \tag{3-28}$$

Similarly, the boundary condition at the free surface is that the pressure is a constant. If only small perturbations from the assumed motion are considered, i. e., ξ is small, then the higher order terms in equation (3-25) may be neglected, yielding

$$\frac{p}{\rho} = -a_t x - \frac{\partial\phi}{\partial t} \quad . \tag{3-29}$$

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*The fluid velocity normal to the surface is $\bar{q} \cdot \frac{\nabla f}{|\nabla f|}$. Since f=0, $\frac{df}{dt} = \frac{d\bar{R}}{dt} \cdot \nabla f + \frac{\partial f}{\partial t} = 0.$ Hence the surface velocity normal to itself is $\frac{dR}{dt} \cdot \frac{\nabla f}{|\nabla f|} = \frac{-\frac{\partial f}{\partial t}}{|\nabla f|}$.

Differentiating equation (3-29) and retaining only the first order terms results in

$$\frac{D}{Dt}\left(\frac{p}{\rho}\right) = -a_t\frac{\partial\xi}{\partial t} - \frac{\partial^2\phi}{\partial t^2} - [\bar{q}\cdot\nabla]\frac{\partial\phi}{\partial t} = 0, \qquad \text{or} \qquad (3\text{-}30)$$

$$\frac{\partial^2\phi}{\partial t^2} + a_t\frac{\partial\xi}{\partial t} = 0\,. \qquad (3\text{-}31)$$

In deriving equations (3-29) and (3-31) it has been assumed that the x axis of the moving frame is aligned with the unperturbed flight path. In this situation only the x component of $\bar{a}_t$ does not vanish.

There also exists a kinematical relation, namely

$$q_x = \frac{\partial\xi}{\partial t} = \frac{\partial\phi}{\partial x} - \nu_x - \omega_y z + \omega_z y\,. \qquad (3\text{-}32)$$

Substituting equation (3-32) into (3-31) gives the final required result,

$$\frac{\partial^2\phi}{\partial t^2} + a_t\frac{\partial\phi}{\partial x} = a_t[\nu_x + \omega_y z - \omega_z y]\,. \qquad (3\text{-}33)$$

Furthermore, this condition may be applied at the unperturbed free surface position, x = -h. This may be shown by observing a Taylor's expansion of the form,

$$\frac{\partial\phi}{\partial x}(-h+\xi, y, z, t) = \frac{\partial\phi}{\partial x}(-h, y, z, t) + \xi\frac{\partial^2\phi}{\partial x^2}(-h, y, z, t) \qquad (3\text{-}34)$$

$$+ \frac{1}{2}\xi^2\frac{\partial^3\phi}{\partial x^3}(-h, y, z, t) + \text{higher order terms,}$$

and noting again that ξ is a small quantity.

4. Summary of the General Problem

To find the forces and moments excited by a moving tank one first determines the flow pattern using equations:

$$\nabla^2 \phi = 0, \tag{3-26}$$

$$\nabla\phi \cdot \nabla f = \bar{\omega} \times \bar{R} \cdot \nabla f + \bar{\nu} \cdot \nabla f - \frac{\partial f}{\partial t} \tag{3-28}$$

$$\frac{\partial^2 \phi}{\partial t^2} + a_t \frac{\partial \phi}{\partial x} = a_t [\nu_x + \omega_y z - \omega_z y] \tag{3-33}$$

The local pressure may then be found from the equation

$$\frac{p}{\rho} = -a_t x - \frac{\partial \phi}{\partial t} \tag{3-29}$$

Properly weighted integration then yields the required forces and moments.

5. An Example

A. The Flow Pattern

As an example, we now consider the case of a perfectly rigid, cylindrical tank of circular cross-section which is simultaneously undergoing pitching and variation in the angle of attack. This is the same problem as that of ref (8) except that pitching motion is now permitted. We note that, because of symmetry, the yawing, sideslipping tank presents a similar problem.

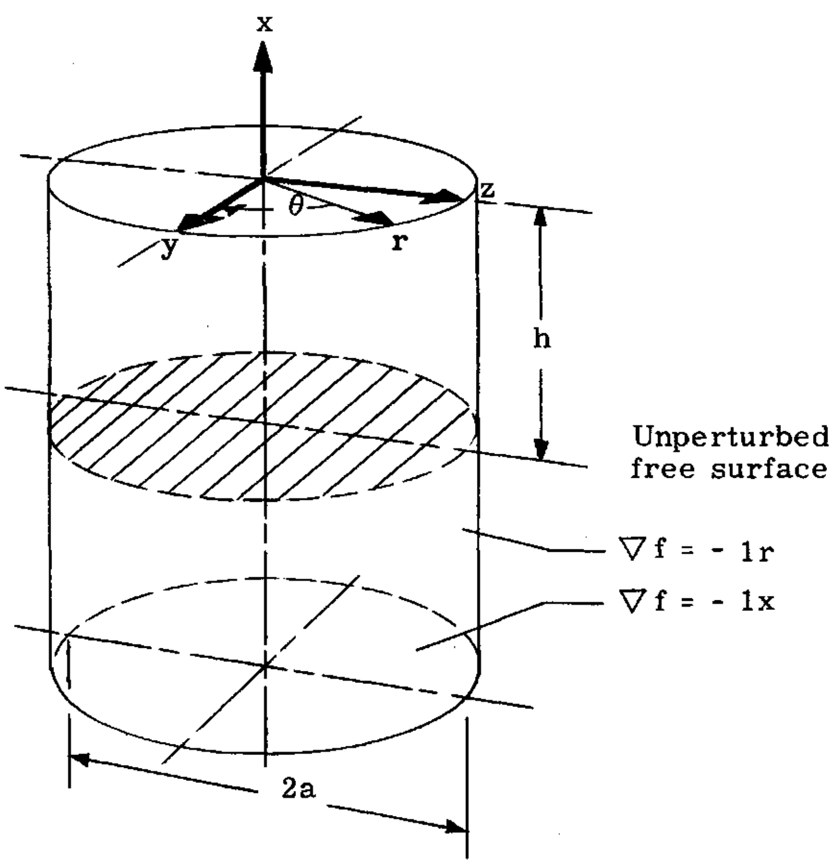

Fig. 1 Example

Noting that

$$\nu_y = \nu_x = \omega_x = \omega_z = 0 \, , \tag{5-1}$$

we may formulate the required boundary conditions. On the tank bottom, i.e., at $x = -h - \frac{\dot{m}}{\pi \rho a^2}$, equation (3-28) becomes

$$\frac{\partial \phi}{\partial x} = \omega_y z = \omega_y r \sin \theta \ . \tag{5-2}$$

On the tank wall, at $\mathbf{r} = a$, we have

$$\frac{\partial \phi}{\partial r} = [\bar{1}_y \cos \theta + \bar{1}_z \sin \theta] \cdot [\omega_y z \bar{1}_x - \omega_y x \bar{1}_z + \nu_z \bar{1}_z] \ . \tag{5-3}$$

After simplification, equation (5-3) is

$$\frac{\partial \phi}{\partial r} = [-\omega_y x + \nu_z] \sin \theta \ . \tag{5-4}$$

In cylindrical coordinates, Laplace's equation (equation [3-26]) is

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial x^2} = 0 \, . \tag{5-5}$$

In order to use arbitrary inputs, it is convenient to take the Laplace transform of the above equations. Assuming that all initial conditions are zero, e.g, $\nu_z \ (t = 0) = 0$, it follows that

$$\frac{\partial^2 \bar{\bar{\phi}}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{\bar{\phi}}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \bar{\bar{\phi}}}{\partial \theta^2} + \frac{\partial^2 \bar{\bar{\phi}}}{\partial x^2} = 0 \ . \tag{5-6}$$

$$\frac{\partial \bar{\bar{\phi}}}{\partial x} = \bar{\bar{\omega}}_y r \sin \ \theta, \ x = -h - \frac{m}{\pi \rho a^2}, \tag{5-7}$$

$$\frac{\partial \bar{\bar{\phi}}}{\partial r} = [-\bar{\bar{\omega}}_y x + \bar{\bar{\nu}}_z] \sin\theta \ , r = a \ , \qquad (5\text{-}8)$$

and from (3-33)

$$s^2 \bar{\bar{\phi}} + a_t \frac{\partial \dot{\bar{\bar{\phi}}}}{\partial x} = a_t \bar{\bar{\omega}}_y r \sin\theta, \ x = -h \ , \qquad (5\text{-}9)$$

where the double bar indicates the transformed quantity.

To solve equation (5-6) a product solution of the form,

$$\bar{\bar{\phi}} = X(x)\, R(r)\, T(\theta) \ , \qquad (5\text{-}10)$$

is assumed. When equation (5-10) is substituted into equation (5-6), the resulting equation is

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -\frac{1}{R} \frac{d^2 R}{dr^2} - \frac{1}{Rr} \frac{dR}{dr} - \frac{1}{r^2 T} \frac{d^2 T}{d\theta^2} = k^2 \ . \qquad (5\text{-}11)$$

It follows from this that X is given by

$$X = C_1 \sinh(kx) + C_2 \cosh(kx) \ , \qquad k \neq 0 \qquad (5\text{-}12)$$

$$X = C_3 x + C_4 \ , \qquad k = 0 \qquad (5\text{-}13)$$

Similarly,

$$\frac{r^2}{R} \frac{d^2 R}{dr^2} + \frac{r}{R} \frac{dR}{dr} + k^2 r^2 = -\frac{1}{T} \frac{d^2 T}{d\theta^2} = n^2, \qquad (5\text{-}14)$$

and therefore

$$T = C_5 \cos(n\theta) + C_6 \sin(n\theta) \ , \qquad (5\text{-}15)$$

and

$$R = C_7 J_n(kr) + C_8 Y_n(kr), \qquad k \neq 0 \qquad (5\text{-}16)$$

$$R = C_9 r^n + C_{10} r^{-n} \ , \qquad k = 0 \ . \qquad (5\text{-}17)$$

The most general solution is a superposition of the $k = 0$ and $k \neq 0$ solutions.

$$\bar{\bar{\phi}} = \left[C_5 \cos(n\theta) + C_6 \sin(n\theta)\right] \left\{\left[C_3 x + C_4\right]\left[C_9 r^n + C_{10} r^{-n}\right]\right.$$

$$\left. + \left[C_1 \sinh(kx) + C_2 \cosh(kx)\right]\left[C_7 J_n(kr) + C_8 Y_n(kr)\right]\right\}. \qquad (5\text{-}18)$$

In order that equation (5-18) satisfy equation (5-8), we have

$$\frac{\partial \bar{\bar{\phi}}}{\partial r} = \left[C_5 \cos(n\theta) + C_6 \sin(n\theta)\right] \left\{\left[C_3 x + C_4\right] n\left[C_9 a^{n-1} - C_{10} a^{-n-1}\right]\right.$$

$$\left. + \left[C_1 \sinh(kx) + C_2 \cosh(kx)\right]\left[C_7 J_n'(ka) + C_8 Y_n'(ka)\right]\right\}$$

$$= \left[-\bar{\bar{\omega}}_y x + \bar{\bar{\nu}}_z\right] \sin\theta \ . \qquad (5\text{-}19)$$

It is apparent that $C_5 = C_{10} = C_8 = 0$; the latter two constants must vanish in order that $\frac{\partial \bar{\bar{\phi}}}{\partial r}$ will be finite at $r = 0$. Also,

$$n = 1 \qquad (5\text{-}20)$$

$$C_6 C_9 C_3 = -\bar{\bar{\omega}}_y \qquad (5\text{-}21)$$

$$C_6 C_9 C_4 = \bar{\bar{\nu}}_z \qquad (5\text{-}22)$$

$$J_1'(ka) = 0 \qquad (5\text{-}23)$$

We denote the roots of equation (5-23) by λ_n:

$$k = \frac{\lambda_n}{a} \ . \qquad (5\text{-}24)$$

Ref(8) gives

$$\lambda_1 = 1.84 ,$$

$$\lambda_2 = 5.335 , \tag{5-25}$$

$$\lambda_3 = 8.535 , \qquad \text{etc.}$$

The general relation for the potential is now

$$\bar{\bar{\phi}} = \sin\theta \left\{ r[-\bar{\bar{\omega}}_y x + \bar{\bar{\nu}}_z] + \sum_{n=1}^{\infty} J_1\left(\lambda_n \frac{r}{a}\right)\left[C_{n_1} \sinh\left(\lambda_n \frac{x}{a}\right) + C_{n_2} \cosh\left(\lambda_n \frac{x}{a}\right)\right]\right\}. \tag{5-26}$$

We now apply the boundary condition at the tank bottom, equation (5-7):

$$\frac{\partial \bar{\bar{\phi}}}{\partial x} = \bar{\bar{\omega}}_y r \sin\theta = \sin\theta \left\{ -\bar{\bar{\omega}}_y r + \sum_{n=1}^{\infty} \frac{\lambda_n}{a} J_1\left(\lambda_n \frac{r}{a}\right)\left[C_{n_1} \cosh \frac{\lambda_n}{a}\left(h + \frac{m}{\pi \rho a^2}\right) - C_{n_2} \sinh \frac{\lambda_n}{a}\left(h + \frac{m}{\pi\rho a^2}\right)\right]\right\}, \tag{5-27}$$

$$2\bar{\bar{\omega}}_y r = \sum_{n=1}^{\infty} \frac{\lambda_n}{a} J_1\left(\lambda_n \frac{r}{a}\right)\left[C_{n_1} \cosh \frac{\lambda_n}{a}\left(h + \frac{m}{\pi\rho a^2}\right) - C_{n_2} \sinh \frac{\lambda_n}{a}\left(h + \frac{m}{\pi\rho a^2}\right)\right]. \tag{5-28}$$

We now use the orthogonality property of two eigenfunctions to find a relation between C_{n_1} and C_{n_2}. By noting certain useful relations in ref(5) and ref(7), it is found that

$$\int_0^a r J_1\left(\lambda_i \frac{r}{a}\right) J_1\left(\lambda_j \frac{r}{a}\right) dr = \delta_{ij} \frac{1}{2}\left(\frac{a}{\lambda_i}\right)^2 (\lambda_i^2 - 1)\, J_1^2(\lambda_i) , \tag{5-29}$$

where δ_{ij} is Kronecker's delta.

It is also found that

$$\int_0^a r^2 J_1\left(\lambda_i \frac{r}{a}\right) dr = \frac{a^3}{\lambda_i{}^2} \; J_i\,(\lambda_i)\;. \tag{5-30}$$

When these results are applied to equation (5-28), it follows that

$$\begin{aligned} 2\bar{\bar{\omega}}_y \frac{a^3}{\lambda_n{}^2} \; J_1(\lambda_n) &= \left[\frac{1}{2}\left(\frac{a}{\lambda_n}\right)\left(\lambda_n{}^2 - 1\right) J_1{}^2(\lambda_n)\right] \cdot \\ &\left[C_{n_1} \cosh \frac{\lambda_n}{a}\left(h + \frac{m}{\pi \rho a^2}\right) - C_{n_2} \sinh \frac{\lambda_n}{a}\left(h + \frac{m}{\pi \rho a^2}\right)\right] \;, \end{aligned} \tag{5-31}$$

or

$$C_{n_2} = C_{n_1} \coth \frac{\lambda_n}{a}\left(h + \frac{m}{\pi \rho a^2}\right) - \frac{4a^2 \bar{\bar{\omega}}_y}{\lambda_n(\lambda_n{}^2 - 1) J_1(\lambda_n) \sinh \frac{\lambda_n}{a}\left(h + \frac{m}{\pi \rho a^2}\right)} \;. \tag{5-32}$$

The potential is now

$$\begin{aligned} \bar{\bar{\phi}} &= \sin\theta \left\{ r\,(-\bar{\bar{\omega}}_y x + \bar{\bar{\nu}}_z) \right. \\ &+ \sum_{n=1}^{\infty} J_1\left(\lambda_n \frac{r}{a}\right) \left[C_{n_1} \sinh\left(\lambda_n \frac{x}{a}\right) + C_{n_1} \coth \frac{\lambda_n}{a}\left(h + \frac{m}{\pi \rho a^2}\right) \cosh\left(\lambda_n \frac{x}{a}\right) \right. \\ &\left. \left. - \frac{4a^2 \bar{\bar{\omega}}_y \cosh\left(\lambda_n \frac{x}{a}\right)}{\lambda_n(\lambda_n{}^2 - 1) J_1(\lambda_n) \sinh \frac{\lambda_n}{a}\left(h + \frac{m}{\pi \rho a^2}\right)} \right] \right\} \;. \end{aligned} \tag{5-33}$$

The final boundary condition, equation (5-9), specifies conditions on the free surface. When equation (5-33) is substituted into equation (5-9) and rearranged the result is

$$s^2 r(\bar{\bar{\omega}}_y h+\bar{\bar{\nu}}_z)-2a_t r\bar{\bar{\omega}}_y \tag{5-34}$$

$$+s^2\sum_{n=1}^{\infty}\frac{J_1\left(\lambda_n \frac{r}{a}\right)}{\sinh \frac{\lambda_n}{a}\left(h+\frac{m}{\pi\rho a^2}\right)}\left[C_{n_1}\cosh\left(\frac{\lambda_n m}{\pi\rho a^3}\right)-\frac{4a^2\bar{\bar{\omega}}_y\cosh\left(\lambda_n\frac{h}{a}\right)}{\lambda_n(\lambda_n^2-1)J_1(\lambda_n)}\right]$$

$$+a_t\sum_{n=1}^{\infty}\frac{\frac{\lambda_n}{a}J_1\left(\lambda_n \frac{r}{a}\right)}{\sinh \frac{\lambda_n}{a}\left(h+\frac{m}{\pi\rho a^2}\right)}\left[C_{n_1}\sinh\left(\frac{\lambda_n m}{\pi\rho a^3}\right)+\frac{4a^2\bar{\bar{\omega}}_y\sinh\left(\lambda_n\frac{h}{a}\right)}{\lambda_n(\lambda_n^2-1)J_1(\lambda_n)}\right]=0,$$

or

$$r\left[2a_t\bar{\bar{\omega}}_y-s^2(\bar{\bar{\omega}}_y h+\bar{\bar{\nu}}_z)\right]= \tag{5-35}$$

$$\sum_{n=1}^{\infty}\frac{J_1\left(\lambda_n \frac{r}{a}\right)}{\sinh \frac{\lambda_n}{a}\left(h+\frac{m}{\pi\rho a^2}\right)}\left\{C_{n_1}\left[s^2\cosh\left(\frac{\lambda_n m}{\pi\rho a^3}\right)+a_t\frac{\lambda_n}{a}\sinh\left(\frac{\lambda_n m}{\pi\rho a^3}\right)\right]\right.$$

$$\left.+\frac{4a^2\bar{\bar{\omega}}_y}{\lambda_n(\lambda_n^2-1)J_1(\lambda_n)}\left[a_t\frac{\lambda_n}{a}\sinh\left(\lambda_n\frac{h}{a}\right)-s^2\cosh\left(\lambda_n\frac{h}{a}\right)\right]\right\}$$

Again the orthogonal properties of the eigenfunctions are used. The resulting equation for C_{n_1} is

$$\frac{a^3}{\lambda_n^2}J_1(\lambda_n)\left[2a_t\bar{\bar{\omega}}_y-s^2(\bar{\bar{\omega}}_y h+\bar{\bar{\nu}}_z)\right]= \tag{5-36}$$

$$\frac{\frac{1}{2}\left(\frac{a}{\lambda_n}\right)^2(\lambda_n^2-1)J_1^2(\lambda_n)}{\sinh\frac{\lambda_n}{a}\left(h+\frac{m}{\pi\rho a^2}\right)}\left\{C_{n_1}\left[s^2\cosh\left(\frac{\lambda_n m}{\pi\rho a^3}\right)+a_t\frac{\lambda_n}{a}\sinh\left(\frac{\lambda_n m}{\pi\rho a^3}\right)\right]\right.$$

$$\left.+\frac{4a^2\bar{\bar{\omega}}_y}{\lambda_n(\lambda_n^2-1)J_1(\lambda_n)}\left[a_t\frac{\lambda_n}{a}\sinh\left(\lambda_n\frac{h}{a}\right)-s^2\cosh\left(\lambda_n\frac{h}{a}\right)\right]\right\}.$$

Hence,

$$Cn_1 = \frac{2a}{(\lambda_n{}^2-1)J_1(\lambda_n)\left[s^2\cosh\left(\frac{\lambda_n m}{\pi\rho a^3}\right) \quad a_t\frac{\lambda_n}{a}\sinh\left(\frac{\lambda_n m}{\pi\rho a^3}\right)\right]} \tag{5-37}$$

$$\times \left\{ \left[\, 2a_t\,\bar{\bar{\omega}}_y - s^2\left(\bar{\bar{\omega}}_y h + \bar{\bar{\nu}}_z\right)\right] \sinh\frac{\lambda_n}{a}\left(h + \frac{m}{\pi\rho a^2}\right)\right.$$

$$\left. - \frac{2a\bar{\bar{\omega}}_y}{\lambda_n}\left[a_t\frac{\lambda_n}{a}\sinh\left(\lambda_n\frac{h}{a}\right) - s^2\cosh\left(\lambda_n\frac{h}{a}\right)\right]\right\} .$$

The complete velocity potential is now found by substitution. It is

$$\bar{\bar{\phi}} = \sin\theta\left\{ r\left(-\bar{\bar{\omega}}_y x + \bar{\bar{\nu}}_z\right)\right. \tag{5-38}$$

$$+\sum_{n=1}^{\infty}\frac{2aJ_1\left(\lambda_n\frac{r}{a}\right)}{(\lambda_n{}^2-1)J_1(\lambda_n)\sinh\frac{\lambda_n}{a}\left(h+\frac{m}{\pi\rho a^2}\right)}\left[-2\frac{a}{\lambda_n}\bar{\bar{\omega}}_y\cosh\left(\lambda_n\frac{x}{a}\right)\right.$$

$$+ \frac{\cosh\frac{\lambda_n}{a}\left(x+h \quad \frac{m}{\pi\rho a^2}\right)}{s^2\cosh\left(\frac{\lambda_n m}{\pi\rho a^3}\right)+a_t\frac{\lambda_n}{a}\sinh\left(\frac{\lambda_n m}{\pi\rho a^3}\right)}\left(2\bar{\bar{\omega}}_y\left[s^2\frac{a}{\lambda_n}\cosh\left(\lambda_n\frac{h}{a}\right) - a_t\sinh\left(\lambda_n\frac{h}{a}\right)\right]\right.$$

$$\left.\left.\left. + \left[2a_t\bar{\bar{\omega}}_y - s^2\left(\bar{\bar{\omega}}_y h + \bar{\bar{\nu}}_z\right)\right]\sinh\frac{\lambda_n}{a}\left(h+\frac{m}{\pi\rho a^2}\right)\right)\right]\right\} .$$

B. Forces and Moments

Inasmuch as the forces and moments exerted by the fluid on the tank walls are found by integrating the pressure, an inspection of equation (3-29) is in order. It is clear that the $a_t x$ term causes no sidewise difference in pressure, and hence no net force or moment. Thus we are led to the working form of equation (3-29),

$$\bar{\bar{p}} = -\rho s \bar{\bar{\phi}} . \tag{5-39}$$

Referring to Fig. 1., it can be seen that the net force in the positive z direction is given by the integral,

$$\bar{\bar{F}}_z = \int_{-h-\frac{m}{\pi \rho a^2}}^{-h} \int_0^{2\pi} \bar{\bar{p}} a \sin\theta \, d\theta \, dx \quad , \tag{5-40}$$

the integration being taken over that part of the tank wall which is immersed. Substituting equations (5-38) and (5-39) into equation (5-40) gives

$$\bar{\bar{F}}_z = \rho a s \int_{-h}^{-h-\frac{m}{\pi \rho a^2}} \int^{2\pi} \bar{\bar{\phi}} \sin\theta \, d\theta \, dx \quad , \tag{5-41}$$

or

$$\bar{\bar{F}}_z = \pi\rho a^2 s \int_{-h}^{-h-\frac{m}{\pi\rho a^2}} \Bigg\{ (-\bar{\bar{\omega}}_y x + \bar{\bar{\nu}}_z) \tag{5-42}$$

$$+ \sum_{n=1}^{\infty} \frac{2}{(\lambda_n^2-1)\sinh\frac{\lambda_n}{a}\left(h+\frac{m}{\pi\rho a^2}\right)} \Bigg[-2\frac{a}{\lambda_n}\bar{\bar{\omega}}_y \cosh\left(\lambda_n\frac{x}{a}\right)$$

$$+ \frac{\cosh\frac{\lambda_n}{a}\left(x+h+\frac{m}{\pi\rho a^2}\right)}{s^2\cosh\left(\frac{\lambda_n m}{\pi\rho a^3}\right)+a_t\frac{\lambda_n}{a}\sinh\left(\frac{\lambda_n m}{\pi\rho a^3}\right)} \Bigg(2\bar{\bar{\omega}}_y\left[s^2\frac{a}{\lambda_n}\cosh\left(\lambda_n\frac{h}{a}\right)-a_t\sinh\left(\lambda_n\frac{h}{a}\right)\right]$$

$$+ \sinh\frac{\lambda_n}{a}\left(h+\frac{m}{\pi\rho a^2}\right)\left[2a_t\bar{\bar{\omega}}_y - s^2(\bar{\bar{\omega}}_y h + \bar{\bar{\nu}}_z)\right]\Bigg)\Bigg]\Bigg\}\, dx.$$

When the integration is finally carried out, the z component of force is

$$\bar{\bar{F}}_z = \pi\rho a^2 s \Bigg\{ \frac{-m}{\pi\rho a^2}\left[\bar{\bar{\nu}}_z + \bar{\bar{\omega}}_y\left(h+\frac{1}{2}\frac{m}{\pi\rho a^2}\right)\right] \tag{5-43}$$

$$+\sum_{n=1}^{\infty} \frac{2a}{\lambda_n(\lambda_n^2-1)\sinh\frac{\lambda_n}{a}\left(h+\frac{m}{\pi\rho a^2}\right)} \Bigg[\frac{4a}{\lambda_n}\bar{\bar{\omega}}_y \sinh\left(\frac{\lambda_n m}{2\pi\rho a^3}\right)\cosh\frac{\lambda_n}{a}\left(h+\frac{1}{2}\frac{m}{\pi\rho a^2}\right)$$

$$+ \frac{\tanh\left(\frac{\lambda_n m}{\pi \rho a^3}\right)}{s^2 + a_t \frac{\lambda_n}{a} \tanh \frac{\lambda_n m}{\pi \rho a^3}} \left(2\bar{\bar{\omega}}_y \left[s^2 \frac{a}{\lambda_n} \cosh(\lambda_n \tfrac{h}{a}) - a_t \sinh(\lambda_n \tfrac{h}{a}) \right] \right.$$

$$\left. + [2a_t \bar{\bar{\omega}}_y - s^2(\bar{\bar{\omega}}_y h + \bar{\bar{\nu}}_z)] \sinh \frac{\lambda_n}{a}\left(h + \frac{m}{\pi \rho a^2}\right) \right) \Big] \Bigg\} . \qquad (5\text{-}43) \text{ cont'd.}$$

In a similar fashion the moment is found to be

$$\bar{\bar{M}}_y = - \int_{-h - \frac{m}{\pi \rho a^2}}^{-h} \int_0^{2\pi} \bar{\bar{p}} a x \sin\theta \, d\theta \, dx \qquad (5\text{-}44)$$

$$- \int_0^a \int_0^{2\pi} \bar{\bar{p}} z \, r \, dr \, d\theta \quad ,$$

where the first integral represents the contribution of the tank wall and the second that of the tank bottom. Using equations (5-38) and (5-39) as before gives the moment:

$$\bar{\bar{M}}_y = \pi \rho a^2 s \int_{-h - \frac{m}{\pi \rho a^2}}^{-h} (-\bar{\bar{\omega}}_y x + \bar{\bar{\nu}}_z) \qquad (5\text{-}45)$$

$$+\sum_{n=1}^{\infty}\frac{2}{(\lambda_n^{2}-1)\sinh\frac{\lambda_n}{a}\left(h+\frac{m}{\pi\rho a^2}\right)}\left[-2\frac{a}{\lambda_n}\bar{\bar{\omega}}_y\cosh\left(\lambda_n\frac{x}{a}\right)\right.$$ (5-45) cont'd.

$$+\frac{\cosh\frac{\lambda_n}{a}\left(x+h+\frac{m}{\pi\rho a^2}\right)}{s^2\cosh\left(\frac{\lambda_n m}{\pi\rho a^3}\right)+a_t\frac{\lambda_n}{a}\sinh\left(\frac{\lambda_n m}{\pi\rho a^3}\right)}\left(2\bar{\bar{\omega}}_y\left[s^2\frac{a}{\lambda_n}\cosh\left(\lambda_n\frac{h}{a}\right)-a_t\sinh\left(\lambda_n\frac{h}{a}\right)\right]\right.$$

$$\left.\left.+\left[2a_t\bar{\bar{\omega}}_y-s^2\left(\bar{\bar{\omega}}_y h+\bar{\bar{\nu}}_z\right)\right]\sinh\frac{\lambda_n}{a}\left(h+\frac{m}{\pi\rho a^2}\right)\right)\right]\Bigg\}\ xdx$$

$$+\pi\rho s\int_0^a\Bigg\{r\left[\bar{\bar{\omega}}_y\left(h+\frac{m}{\pi\rho a^2}\right)+\bar{\bar{\nu}}_z\right]$$

$$+\sum_{n=1}^{\infty}\frac{2a\,J_1\left(\lambda_n\frac{r}{a}\right)}{(\lambda_n^{2}-1)J_1(\lambda_n)\sinh\frac{\lambda_n}{a}\left(h+\frac{m}{\pi\rho a^2}\right)}\left[-2\frac{a}{\lambda_n}\bar{\bar{\omega}}_y\cosh\frac{\lambda_n}{a}\left(h+\frac{m}{\pi\rho a^2}\right)\right.$$

$$+\frac{1}{s^2\cosh\left(\frac{\lambda_n m}{\pi\rho a^3}\right)+a_t\frac{\lambda_n}{a}\sinh\left(\frac{\lambda_n m}{\pi\rho a^3}\right)}\left(2\bar{\bar{\omega}}_y\left[s^2\frac{a}{\lambda_n}\cosh\left(\lambda_n\frac{h}{a}\right)-a_t\sinh\left(\lambda_n\frac{h}{a}\right)\right]\right.$$

$$\left.\left.+\left[2a_t\bar{\bar{\omega}}_y-s^2\left(\bar{\bar{\omega}}_y h+\bar{\bar{\nu}}_z\right)\right]\sinh\frac{\lambda_n}{a}\left(h+\frac{m}{\pi\rho a^2}\right)\right)\right]\Bigg\}\ r^2dr$$

When the integrals in equation (5-45) are evaluated, it is found that

$$\bar{\bar{M}}_y=\pi\rho a^2 s\Bigg\{\frac{-m}{\pi\rho a^2}\left[\bar{\bar{\omega}}_y\left(h^2+h\frac{m}{\pi\rho a^2}+\frac{1}{3}\left(\frac{m}{\pi\rho a^2}\right)^2\right)\right. \qquad (5\text{-}46)$$

$$+\bar{\bar{\nu}}_z\left(h+\frac{1}{2}\frac{m}{\pi\rho a^2}\right)\Bigg]+\frac{a^2}{4}\left[\bar{\bar{\omega}}_y\left(h+\frac{m}{\pi\rho a^2}\right)+\bar{\bar{\nu}}_z\right] \tag{5-46 cont'd.}$$

$$+\sum_{n=1}^{\infty}\frac{2a^2}{\lambda_n^2(\lambda_n^2-1)\sinh\frac{\lambda_n}{a}\left(h+\frac{m}{\pi\rho a^2}\right)}\left[\left(\frac{-\frac{\lambda_n h}{a}\tanh\left(\frac{\lambda_n m}{\pi\rho a^3}\right)+2\operatorname{sech}\left(\frac{\lambda_n m}{\pi\rho a^3}\right)-1}{s^2+a_t\frac{\lambda_n}{a}\tanh\left(\frac{\lambda_n m}{\pi\rho a^3}\right)}\right)\right.$$

$$\left(\left[2a_t\bar{\bar{\omega}}_y-s^2(\bar{\bar{\omega}}_y h+\bar{\bar{\nu}}_z)\right]\sinh\frac{\lambda_n}{a}\left(h+\frac{m}{\pi\rho a^2}\right)+2\bar{\bar{\omega}}_y\left[s^2\frac{a}{\lambda_n}\cosh\left(\lambda_n\frac{h}{a}\right)\right.\right.$$

$$\left.\left.-a_t\sinh\left(\lambda_n\frac{h}{a}\right)\right]\right)+4\frac{a}{\lambda_n}\bar{\bar{\omega}}_y\left(\frac{\lambda_n}{2a}\left(h+\frac{m}{\pi\rho a^2}\right)\sinh\frac{\lambda_n}{a}\left(h+\frac{m}{\pi\rho a^2}\right)\right.$$

$$\left.\left.\left.+\frac{1}{2}\cosh\left(\lambda_n\frac{h}{a}\right)-\frac{\lambda_n h}{2a}\sinh\left(\lambda_n\frac{h}{a}\right)-\cosh\frac{\lambda_n}{a}\left(h+\frac{m}{\pi\rho a^2}\right)\right)\right]\right\}$$

In order to use equations (5-43) and (5-46) in missile dynamics work, it is convenient to write the above results in terms of four transfer functions:

$$\frac{\bar{\bar{F}}_z}{\bar{\bar{\nu}}_z}=-\pi\rho a^2 s\left\{\frac{m}{\pi\rho a^2}+\sum_{n=1}^{\infty}\frac{2as^2\tanh\left(\frac{\lambda_n m}{\pi\rho a^3}\right)}{\lambda_n(\lambda_n^2-1)\left[s^2+a_t\frac{\lambda_n}{a}\tanh\left(\frac{\lambda_n m}{\pi\rho a^3}\right)\right]}\right\} \tag{5-47}$$

$$\frac{\bar{\bar{F}}_z}{\bar{\bar{\omega}}_y}=\pi\rho a^2 s\left\{-\frac{m}{\pi\rho a^2}\left(h+\frac{1}{2}\frac{m}{\pi\rho a^2}\right)\right. \tag{5-48}$$

$$+\sum_{n=1}^{\infty}\frac{2a}{\lambda_n(\lambda_n^2-1)\sinh\frac{\lambda_n}{a}\left(h+\frac{m}{\pi\rho a^2}\right)}\left[4\frac{a}{\lambda_n}\sinh\left(\frac{\lambda_n m}{2\pi\rho a^3}\right)\cosh\frac{\lambda_n}{a}\left(h+\frac{1}{2}\frac{m}{\pi\rho a^2}\right)\right.$$

$$+\frac{\tanh\frac{\lambda_n m}{\pi\rho a^3}}{s^2+a_t\frac{\lambda_n}{a}\tanh\left(\frac{\lambda_n m}{\pi\rho a^3}\right)}\left(\frac{2a}{\lambda_n}\cosh\left(\lambda_n\frac{h}{a}\right)\left[s^2-a_t\frac{\lambda_n}{a}\tanh\left(\lambda_n\frac{h}{a}\right)\right]\right.$$

$$\left.+\left[2a_t-s^2h\right]\sinh\frac{\lambda_n}{a}\left(h+\frac{m}{\pi\rho a^2}\right)\right)\Bigg]\Bigg\} \qquad \text{(5-48) cont'd.}$$

$$\frac{\bar{\bar{M}}_y}{\bar{\bar{\nu}}_z}=\pi\rho a^2 s\left\{\frac{a^2}{4}+\left(h+\frac{1}{2}\frac{m}{\pi\rho a^2}\right)\right. \qquad \text{(5-49)}$$

$$\left.+\sum_{n=1}^{\infty}\frac{2a^2s^2\left[\frac{\lambda_n h}{a}\tanh\left(\frac{\lambda_n m}{\pi\rho a^3}\right)-2\operatorname{sech}\left(\frac{\lambda_n m}{\pi\rho a^3}\right)+1\right]}{\lambda_n^2(\lambda_n^2-1)\left[s^2+a_t\frac{\lambda_n}{a}\tanh\left(\frac{\lambda_n m}{\pi\rho a^3}\right)\right]}\right\}$$

$$\frac{\bar{\bar{M}}_y}{\bar{\bar{\omega}}_y}=\pi\rho a^2 s\left\{\frac{-m}{\pi\rho a^2}\left(h^2+h\frac{m}{\pi\rho a^2}+\frac{1}{3}\left(\frac{m}{\pi\rho a^2}\right)^2\right)\right. \qquad \text{(5-50)}$$

$$+\frac{a^2}{4}\left(h+\frac{m}{\pi\rho a^2}\right)+\sum_{n=1}^{\infty}\frac{2a^2}{\lambda_n^2(\lambda_n^2-1)\sinh\frac{\lambda_n}{a}\left(h+\frac{m}{\pi\rho a^2}\right)}\left[\left(\frac{-\frac{\lambda_n h}{a}\tanh\left(\frac{\lambda_n m}{\pi\rho a^3}\right)+2\operatorname{sech}\left(\frac{\lambda_n m}{\pi\rho a^3}\right)-1}{s^2+a_t\frac{\lambda_n}{a}\tanh\left(\frac{\lambda_n m}{\pi\rho a^3}\right)}\right)\right.$$

$$\left(\left[2a_t-s^2h\right]\sinh\frac{\lambda_n}{a}\left(h+\frac{m}{\pi\rho a^2}\right)+2\left[s^2\frac{a}{\lambda_n}\cosh\left(\lambda_n\frac{h}{a}\right)-a_t\sinh\left(\lambda_n\frac{h}{a}\right)\right]\right)$$

$$+4\frac{a}{\lambda_n}\left(\frac{\lambda_n}{2a}\left(h+\frac{m}{\pi\rho a^2}\right)\sinh\frac{\lambda_n}{a}\left(h+\frac{m}{\pi\rho a^2}\right)+\frac{1}{2}\cosh\left(\lambda_n\frac{h}{a}\right)\right.$$

$$\left.\left.\left.-\frac{\lambda_n h}{2a}\sinh\left(\lambda_n\frac{h}{a}\right)-\cosh\frac{\lambda_n}{a}\left(h+\frac{m}{\pi\rho a^2}\right)\right)\right]\right\}.$$

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